Beilinson–Bernstein localization

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Outline

Throughout, let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ and $X = \mathbb{P}^1$.
We let Ω denote the Casimir element, which generates the centre of $U(\mathfrak{g})$.
In this talk, we give a geometric description of all representations of $\mathfrak{g}$ on which Ω acts trivially. Indeed, these representations arise as the global sections of $\mathcal{D}$-modules on $\mathbb{P}^1$.

Theorem (Beilinson–Bernstein, Brylinski–Kashiwara)

1. *The morphism*
$$U(\mathfrak{g})/\Omega \to \mathcal{D}_X(X)$$
is an isomorphism.
2. *The global section functor is exact on $\mathcal{D}_X$-modules.*
3. *If $\mathcal{M}$ is a $\mathcal{D}_X$-module with $\mathcal{M}(X) = 0$, then $\mathcal{M} = 0$.*

Corollary

The global section functor yields an equivalence of categories between $\mathcal{D}_X$-modules and $U(\mathfrak{g})/\Omega$-modules.
The quasi-inverse is given by localization: $M \mapsto \mathcal{D}_X \otimes_{\mathcal{D}_X(X)} M$.

Proof of the Corollary:
Write Loc for the localization functor. If M is a $\mathcal{D}_X(X)$-module, consider

$$\mathcal{D}_X(X)^{\oplus I} \to \mathcal{D}_X(X)^{\oplus J} \to M \to 0.$$

Since Loc is right exact and the global section functor is exact, the fact that $(\mathrm{Loc} M)(X) \cong M$ follows from the case $M = \mathcal{D}_X(X)$.
It is an easy consequence of 2. and 3. that the global sections functor reflects isomorphisms.
If now $\mathcal{M}$ is a $\mathcal{D}_X$-module, consider the natural morphism $\mathrm{Loc}(\mathcal{M}(X)) \to \mathcal{M}$. This is an isomorphism on global sections by the above, hence an isomorphism.

If A is a $\mathbb{C}$-algebra, an $\mathbb{N}$-filtration on A is a sequence of subspaces

$$\mathrm{Fil}_0 A \subseteq \mathrm{Fil}_1 A \subseteq \ldots \subseteq A$$

such that $1 \in \mathrm{Fil}_0 A$, $\cup \mathrm{Fil}_n A = A$ and $\mathrm{Fil}_i A \cdot \mathrm{Fil}_j A \subseteq \mathrm{Fil}_{i+j} A$.
In this case, the associated graded space

$$\mathrm{gr}\ A := \oplus \mathrm{Fil}_n A / \mathrm{Fil}_{n-1} A$$

inherits the structure of a (graded) $\mathbb{C}$-algebra (set $\mathrm{Fil}_{-1} A = 0$).

Example

1. $U(\mathfrak{g})$ is filtered by polynomial degree and $\mathrm{gr}U(\mathfrak{g}) \cong \mathrm{Sym}\mathfrak{g}$ (PBW).
2. The first Weyl algebra A_1 is filtered by order of differential operators: $\mathrm{Fil}_0 A_1 = \mathbb{C}[x]$, $\mathrm{Fil}_1 A_1 = \mathbb{C}[x] + \mathbb{C}[x]\partial, \ldots$, and $\mathrm{gr}A_1 \cong \mathbb{C}[X, Y]$.
3. More generally: if X is any smooth scheme, then $\mathcal{D}_X(X)$ is filtered with $\mathrm{gr}\mathcal{D}_X(X) \cong \mathcal{O}(T^*X)$.

Lemma

Let A, B be $\mathbb{N}$-filtered algebras and let $f : A \to B$ be a morphism of algebras such that $f(\mathrm{Fil}_i A) \subseteq \mathrm{Fil}_i B$. Then f induces a morphism $\mathrm{gr} f : \mathrm{gr} A \to \mathrm{gr} B$. If $\mathrm{gr} f$ is an isomorphism, then so is f.

We can now prove that the Beilinson–Bernstein map is indeed an isomorphism: it preserves the filtration, and the associated graded map

$$\mathbb{C}[e, h, f]/(h^2 + 4fe) \to \mathcal{O}(T^*\mathbb{P}^1)$$

is indeed an isomorphism.

Let $\mathcal{O}(1)$ be Serre's twisting sheaf on $\mathbb{P}^1$ and $\mathcal{O}(n) = \mathcal{O}(1)^{\otimes n}$ for $n \geq 0$. Note that $\mathcal{O}(n)(\mathbb{P}^1) \cong L(n)$, so there are natural maps $\mathcal{O} \otimes_{\mathbb{C}} L(n) \to \mathcal{O}(n)$ and (after dualizing and tensoring) $\mathcal{O} \to \mathcal{O}(n) \otimes_{\mathbb{C}} L(n)$.
The key point is now that if $\mathcal{M}$ is a $\mathcal{D}$-module on $\mathbb{P}^1$, then the induced epimorphism $\mathcal{M} \otimes_{\mathbb{C}} L(n) \to \mathcal{M} \otimes_{\mathcal{O}} \mathcal{O}(n)$ and the induced monomorphism $\mathcal{M} \to \mathcal{M} \otimes_{\mathcal{O}} \mathcal{O}(n) \otimes_{\mathbb{C}} L(n)$ actually split in the category of sheaves! (All objects carry natural $\mathfrak{g}$-actions, decompose into generalized Ω-eigenspaces.)

The statement of the theorem now follows by using Serre vanishing for all $\mathcal{O}$-coherent $\mathcal{O}$-submodules of $\mathcal{M}$.
E.g., if $\mathcal{N}$ is any coherent submodule of $\mathcal{M}$, then the diagram

$$\begin{array}{ccc} \mathrm{H}^1(X,\mathcal{N}) & \longrightarrow & \mathrm{H}^1(X,\mathcal{M}) \\ \downarrow & & \downarrow \\ 0 = \mathrm{H}^1(X,\mathcal{N}(n)) \otimes L(n) & \longrightarrow & \mathrm{H}^1(X,\mathcal{M}(n)) \otimes L(n) \end{array}$$

for n large shows that $\mathrm{H}^1(X,\mathcal{N}) \to \mathrm{H}^1(X,\mathcal{M})$ is the zero map.
But $\mathrm{H}^1(X,\mathcal{M}) \cong \varinjlim \mathrm{H}^1(X,\mathcal{N})$, showing exactness of global sections.
In our $\mathfrak{sl}_2$-case, there is also a purely $\mathcal{D}$-module theoretic proof, by lifting modules on $\mathbb{P}^1$ to $\mathbb{A}^2 \setminus \{(0,0)\}$.

We have seen some standard $U(\mathfrak{g})$-modules in Tobias' talk. Which $\mathcal{D}$-modules do they correspond to?

Example ($\mathrm{Loc}\, L(0) = \mathcal{O}$)

$\mathcal{O}_{\mathbb{P}^1}$ is a $\mathcal{D}_{\mathbb{P}^1}$-module in a natural way. Its global sections are $\mathcal{O}(\mathbb{P}^1) = \mathbb{C}$ (the constant functions). e, f, h (resp. the corresponding derivations) all act trivially, i.e. $\mathcal{O}(\mathbb{P}^1) \cong L(0)$.

Example ($\mathrm{Loc} M(0)^\vee = j_*\mathcal{O}_U$)

Let $U = \mathrm{Spec}\,\mathbb{C}[t] \subset \mathbb{P}^1$, and let $j : U \to \mathbb{P}^1$ denote the embedding. $(j_*\mathcal{O}_U)(\mathbb{P}^1) = \mathcal{O}(U) = \mathbb{C}[t]$ carries a natural $\mathcal{D}(\mathbb{P}^1)$-module structure, via the map $\mathcal{D}(\mathbb{P}^1) \to \mathcal{D}(U)$.
Julian already calculated that e acts as $-\partial_t$, h as $-2t\partial_t$, f as $t^2\partial_t$.
In particular, $\{t^n : \; n \geq 0\}$ is a basis of h-eigenvectors with eigenvalues $-2n$.

Example ($j_*\mathcal{O}_U$, continued)

Calculating the action of e and f, we find $\mathbb{C}[t] \cong M(0)^\vee$, with the constant functions $\mathbb{C} \cong L(0) \subset \mathbb{C}[t]$ as the unique irreducible submodule. On the level of $\mathcal{D}$-modules, this inclusion corresponds to the natural morphism $\mathcal{O}_X \to j_*\mathcal{O}_U$. Its cokernel has global sections $\mathbb{C}[t]/\mathbb{C}$, which is isomorphic to $M(-2) = L(-2)$.

There is also a notion of a $\mathcal{D}$-module dual, and the localization of $M(0)$ is given by the dual of j_*O_U.